Seismic Modeling, Migration and Velocity Inversion

Finite Difference Approximations of the Wave Equations

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May 30, 2014



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Outline

Finite Differences

- Finite Difference Approximations
- Taylor Series Differences
- Central Differences

2) Two-Way Equations

- Application to the 2D Two-Way Scalar Wave Equation
- Lax-Wendroff or the Dablain Trick
- Application to the 2D Two-Way Scalar Wave Equation
- Summary

3 One-Way Wave Equations

- Application to the One-Way XT Scalar Wave Equation
- Application to the One-Way FX Scalar Wave Equation
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Finite Difference Approximations

Once we have the basic equations we can produce digital propagating equations by simply replacing the various derivatives by central difference formulas. Accuracy is dependent only on the accuracy of the differential approximations. The tremendous literature on such approximations generally falls into two categories:

- Polynomial approximations
 - Fits a polynomial to discrete data values
 - · Uses the derivative of the polynomial to produce a difference formula
- Taylor approximations
 - Uses a Taylor series expansion of functions to produce difference formulas
 - A natural extensions of the differential equations
- Of these two the Taylor series method is by far the most popular
 - It will be the focus of the rest of this section



Taylor Series Differences

The Taylor series for $u(x \pm \Delta x)$ in terms of u(x) is

$$u(x \pm \Delta x) = u(x) \pm \frac{\partial u}{\partial x} \Delta x + \frac{\partial^2 u}{\partial x^2} \frac{\Delta x^2}{2!} \pm \frac{\partial^3 u}{\partial x^3} \frac{\Delta x^3}{3!} + \cdots$$

If we rearrange this series in the form

$$\frac{u(x\pm\Delta x)-u(x)}{\Delta x}=\pm\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x^2}\frac{\Delta x}{2!}\pm\frac{\partial^3 u}{\partial x^3}\frac{\Delta x^2}{3!}+\cdots$$

we immediately recognize that the forward and backward differences are accurate to Δx . Mathematically we say that the forward and backward difference are are on the order of Δx , or just $O(\Delta x)$.

Taylor series form the basis for other more accurate formulas. The most obvious one arises from the sum of the Taylor series expansions for $u(x + \Delta x) - u(x)$ and $u(x) - u(x - \Delta x)$. This immediately yields the central difference formula

$$\frac{u(x+\Delta x)-u(x-\Delta x)}{2\Delta x}=\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\frac{\Delta x^2}{3!}+\frac{\partial^5 u}{\partial x^5}\frac{\Delta x^4}{5!}+\cdots$$

which is $O(\Delta x^2)$. Since we generally think of Δx as being small in magnitude this central difference formula is clearly an improvement over a first-order forward or backward difference.



Extension of the second order central difference to higher orders is tedious, but straight forward. For any given k (real or integer) one has

$$\frac{u(x+k\Delta x)+u(x-k\Delta x)}{2} = u(x)+k^2\frac{\partial u^2}{\partial x^2}\frac{\Delta x^2}{2!}+k^4\frac{\partial^4 u}{\partial x^4}\frac{\Delta x^4}{4!} + k^6\frac{\partial^6 u}{\partial x^6}\frac{\Delta x^6}{6!}+k^8\frac{\partial^8 u}{\partial x^8}\frac{\Delta x^8}{8!}\cdots$$



If we want a fourth order scheme, what we do is take the two terms

$$\begin{array}{lll} u(x + \Delta x) + u(x - \Delta x) &=& 2(u(x) + \frac{\partial^2 u}{\partial x^2} \frac{\Delta x^2}{2!} + \frac{\partial^4 u}{\partial x^4} \frac{\Delta x^4}{4!}) \\ u(x + 2\Delta x) + u(x - 2\Delta x) &=& 2(u(x) + 4\frac{\partial^2 u}{\partial x^2} \frac{\Delta x^2}{2!} + 16\frac{\partial^4 u}{\partial x^4} \frac{\Delta x^4}{4!}) \end{array}$$

solve the second for the fourth order partial derivative and substitute into the first to obtain

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x+2\Delta x)+16u(x+\Delta x)-34u(x)+16u(x-\Delta x)+u(x-2\Delta x)}{12\Delta x^2}$$

Higher order central difference approximations are obtained by simply adding additional terms to the mix. For example, a 10th order accurate term is obtained by back-substitution in the five equations when k = 1, 2, 3, 4, 5. The result is a scheme of the form

$$\frac{\partial^2 u}{\partial x^2} \approx \sum_{k=-5}^{k=5} w_k u(x-k\Delta x)$$

where

| —k— | W |
|-----|---------------|
| 0 | -5.854444444 |
| 1 | 3.3333333333 |
| 2 | -0.4761904762 |
| 3 | 0.0793650794 |
| 4 | -0.0099206349 |
| 5 | 0.0006349206 |

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An Explicit 2D Finite Difference Propagator

Applying the difference approximations to the second-order-scalar wave equation with solution $u_{i,j,n+1} = u(i\Delta x, j\Delta z, n\Delta t + \Delta t)$ yields the 2D discrete central difference formula forward extrapolation

$$u_{i,j,n+1} = 2u_{i,j,n} - u_{i,j,n-1} + v^2 \left(\sum_k b_k u_{i-k,j,n} + \sum_m c_m u_{i,j-m,n} \right) + s_{i_0,j_0,n}$$

for the 2D scalar wave equation, where for clarity the factors Δt^2 , Δx^2 and Δy^2 have been suppressed. Here, $s_{i_0,j_0,n}$ represents a source at the location specified by i_0 and j_0 .





The extrapolator in the previous section is of second order in time and Nth order in space.

Some key points are:

- The extrapolator requires exactly 3 volumes in memory at all times
- Extension to higher orders in time
 - Increases the accuracy, but also increases the number of volumes that must be held in memory
- A natural question is whether or not the time order can be increased
 - Without increasing the number of volumes that must be held in memory
- The answer is the Lax-Wendroff or Dablain Trick



Lax-Wendroff or The Dablain Trick

Probably the best known "trick" for improving derivatives in the time direction was initially published by Lax and Wendroff some 40 years ago. (see also Dablain (1986)) What they did was use the wave equation to find a fourth order accurate difference for $\frac{\partial^2}{\partial l^2}$ that does not increase the overall memory requirements. To understand this trick, consider the case in 2-dimensions when the velocity is constant and $\rho = 1$. If we solve the Taylor series for the simplest 2nd order time differential we get

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\Delta t^2} \left(u(t + \Delta t) - 2u(t) + u(t - \Delta t) - \sum_{i=2}^{i=\infty} \frac{\partial^{2i} u}{\partial t^{2i}} \frac{\Delta t^{2i}}{2i!} \right)$$
$$\approx \frac{1}{\Delta t^2} \left(u(t + \Delta t) - 2u(t) + u(t - \Delta t) - \frac{\partial^4 u}{\partial t^4} \frac{\Delta t^4}{12!} \right)$$

We also know that

$$\frac{\partial^2 u}{\partial t^2} = v^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

S0

$$\begin{aligned} \frac{{}^{4}u}{t^{4}} &= v^{2}\left[\frac{\partial^{2}u}{\partial x^{2}}\left(\frac{\partial^{2}u}{\partial t^{2}}\right) + \frac{\partial^{2}u}{\partial z^{2}}\left(\frac{\partial^{2}u}{\partial t^{2}}\right)\right] \\ &= v^{2}\left[\frac{\partial^{2}u}{\partial x^{2}}\left(\frac{\partial^{2}u}{\partial x^{2}} + \frac{\partial^{2}u}{\partial z^{2}}\right) + \frac{\partial^{2}u}{\partial z^{2}}\left(\frac{\partial^{2}u}{\partial x^{2}} + \frac{\partial^{2}u}{\partial z^{2}}\right)\right] \\ &= v^{4}\left(\frac{\partial^{4}u}{\partial x^{4}} + 2\frac{\partial^{4}u}{\partial x^{2}\partial z^{2}} + \frac{\partial^{4}u}{\partial x^{4}}\right) \end{aligned}$$

which tells us that we can replace the fourth order time differential with spatial derivatives. This means that we can increase the accuracy without increasing memory requirements.

Lax-Wendroff

It should be noted that the assumptions of constant density and velocity are not necessary. What the Lax-Wendroff scheme does is generalizes our scheme for finding higher order central difference terms through the recursive formula

$$\frac{\partial^{2i} u}{\partial t^{2i}} = -\left(\rho v^2 \nabla \cdot \frac{1}{\rho} \nabla u\right) \frac{\partial^{2i-2} u}{\partial t^{2i-2}}$$

The Lax-Wendroff 2D Finite Difference Propagator

Applying the difference approximations to $u_{i,j,n+1} = u(i\Delta x, j\Delta z, n\Delta t + \Delta t)$ produces the 2D discrete central difference formula forward extrapolator

$$u_{i,j,n+1} = 2u_{i,j,n} - u_{i,j,n-1} \\ + v^2 \left(\sum_{k} b_k u_{i-k,j,n} + \sum_{m} c_m u_{i,j-m,n} \right) \\ + v^4 \sum_{k} \sum_{m} a_{k,m} u_{i-k,j-m,n} + s_{i_0,j_0,n}$$

for the 2D scalar wave equation, where for clarity the factors Δt^2 , Δx^2 and Δy^2 have been suppressed. Here, $s_{i_0,j_0,n}$ represents a source at the location specified by i_0 and j_0 .

Finite Difference Approximation Summary

Taylor series

- Represent the typical approach to differential approximation
- Accuracy can be of almost any order
- Experience has shown that 8×4 is the break even point
- Lax-Wendroff or Dablain
 - Approximates time derivatives with spatial derivatives
 - Reduces need for increased memory
- Accuracy
 - Dependent only on accuracy of differential approximates
 - Tends to stabilize around 8th order
 - Always have some inaccuracies



Staggered Grids

The first-order fully elastic equation is usually discretized over what has become known as a staggered grid. Assuming a grid spacing of Δx , the actual calculations take place on two grids that are staggered with respect to one another. Some of the parameters exist on one while other exist on the other. A couple of the parameters actually live on both. What is proposed, then, is two grids. with one centered on *x* and the other on $x + .5\Delta x$. Think of this as two separate screens offset by half the screen spacing.



Staggered Grids

Constructing a staggered grid requires the approximation of derivatives at half-spacings. Certainly this is well within our scope. Approximate derivatives using $\frac{\Delta x}{2}$:

$$\frac{u(x+\frac{\Delta x}{2})-u(x-\frac{\Delta x}{2})}{\Delta x} = u(x) + \frac{\partial u^2}{\partial x^2} \frac{\Delta x^2}{4 \times 2!} + \frac{\partial^4 u}{\partial x^4} \frac{\Delta x^4}{16 \times 4}$$
$$\frac{u(x+\Delta x)-u(x-\Delta x)}{\Delta x} = u(x) + \frac{\partial u^2}{\partial x^2} \frac{\Delta x^2}{2!} + \frac{\partial^4 u}{\partial x^4} \frac{\Delta x^4}{4!}$$

The difference formula then becomes

$$\frac{\partial^2 u}{\partial x^2} \approx \sum_{i=-N}^{i=N} w_i u(x-\frac{i}{2}\Delta x)$$

Staggered Grid Parameters



Distribution of variables and parameters (ρ, c_{ij}) in a 2D staggered grid mesh. Particle velocity v^2 lies on the regular grid, parameters c_{ij} , and σ^{ij} lie on the half grid, v^2 on the full grid and ρ lies on both. Memory conservation is a density benefit of this approach

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2D Staggered Grid Finite Difference Propagator

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A 2D Staggered Grid Propagator at Work



Staggered grid finite difference stencils. Purple shading represents the regular gird so that nodes in the middle lie on the regular grid while those the edges lie on the half grid.

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Summary

Two fundamental discrete equations

- One for scalar equations
 - Central differences on regular grid
- One for elastic equations
 - Staggered grids



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One-Way Scalar Wave Equation in XT

The one-way scalar wave equation in space time is

$$\frac{\partial u}{\partial z} = \pm \sqrt{\left(\frac{1}{V^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right)u}$$

If we set

$$T^2 = \frac{\partial^2 u}{\partial t^2}, Z = \frac{\partial u}{\partial z}, X^2 = \frac{\partial^2 u}{\partial x^2} \text{ and } Y^2 = \frac{\partial^2 u}{\partial y^2}$$

then the scalar wave equation takes the form

$$Z=\pm\sqrt{\frac{T^2}{V^2}-X^2-Y^2}$$

Getting Rid of the Square Root

Approximate the square root:

$$Z = \pm \sqrt{\frac{T^2}{V^2} - X^2 - Y^2} \approx \pm \left(\frac{T}{V} - \frac{4\frac{T^2}{V^2} - 3(X^2 + Y^2)}{4\frac{T^2}{V^2} - (X^2 + Y^2)}\right)$$

Clear fractions:

$$(4\frac{T^2}{V^2} - (X^2 + Y^2))Z = \pm \left(4(\frac{T}{V} - 1)\frac{T^2}{V^2} - (\frac{T}{V} - 3)(X^2 + Y^2)\right)$$

One-Way Propagation in XT

Substitute

$$T = rac{\partial U}{\partial t}, \ T^2 = rac{\partial^2 u}{\partial t^2}, \ Z = rac{\partial u}{\partial z}. \ X^2 = rac{\partial^2 u}{\partial x^2} \ ext{and} \ Y^2 = rac{\partial^2 u}{\partial y^2}$$

back into

$$(4\frac{T^2}{V^2} - (X^2 + Y^2))Z = \pm \left(4(\frac{T}{V} - 1)\frac{T^2}{V^2} - (\frac{T}{V} - 3)(X^2 + Y^2)\right)$$

to get

$$\left(4\frac{1}{V^2}\frac{\partial^2 u}{\partial t^2} - (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial Y^2})\right)\frac{\partial u}{\partial z} = \pm \left(4(\frac{1}{V}\frac{\partial u}{\partial t} - 1)\frac{1}{V^2}\frac{\partial^2 u}{\partial t^2} - (\frac{1}{V}\frac{\partial u}{\partial t} - 3)(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2})\right)$$

and replace all the partial derivatives with difference quotients

One-Way XT Finite Differences

Then organize into matrix form

$$\mathbf{A}u(x, y, z + \Delta z, t) = \mathbf{B}u(x, y, z, t)$$

where **A** and **B** are matrices of coefficients derived from finite differences and the underlying Earth model.

Inverting A

$$u(x, y, z + \Delta z, t) = \mathbf{A}^{-1} \mathbf{B} u(x, y, z, t)$$

to produce an *implicit* propagator that steps down one Δz at a time. Inverting A in 3D is not easy and consequently its a source for additional errors in the propagation.



One-Way Scalar Wave Equation in FX

The one-way scalar wave equation in frequency-space is

$$\frac{\partial u}{\partial z} = \pm \sqrt{\left(\frac{\omega^2}{V^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u}$$

If we set

$$T^2 = \omega^2, Z = \frac{\partial u}{\partial z}, X^2 = \frac{\partial^2 u}{\partial x^2}$$
 and $Y^2 = \frac{\partial^2 u}{\partial y^2}$

then the scalar FX wave equation takes the form

$$Z=\pm\sqrt{rac{T^2}{V^2}+X^2+Y^2}$$

which is essentially identical to the XT one-way-scalar wave equation

FX Finite Difference

Applying the same square root approximation to

$$Z = \pm i \sqrt{\frac{\omega^2}{V^2} + X^2 + Y^2}$$

in the frequency domain results in a matrix formulation

$$u(x, y, z + \Delta z, \omega) = \mathbf{A}^{-1}(\omega)\mathbf{B}(\omega)u(x, y, z, \omega)$$

that is very similar to the space-time domain result. It just happens to be done frequency-by-frequency.

XT and FX Finite Difference Summary

- Both implicit methods
 - They require matrix inversions at each downward or upward step
- In 3D the matrices may be huge
- This approximation of the square root is considered to produce the most accurate of all one-way methods
 - But the matrix inversion in 3D makes implementation difficult



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Stability

The factors of the from

$$\frac{v^2 \Delta t}{\Delta x}, \frac{v^2 \Delta t}{\Delta y}, \text{and } \frac{v^2 \Delta t}{\Delta z}$$

are extremely important.

To assure that the computations are stable we must have

$$\Delta t \leq rac{2}{\pi} \left(rac{\Delta x_{min}}{v_{max}}
ight)$$

where Δx_{min} is the smallest of Δx , Δy , and Δz .

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Boundaries



Realistic seismic simulations generally include procedures for suppressing boundary reflections. Modern approaches begin by surrounding the model with a small number of fake layers. Modified equations for absorbing energy are then applied layer by layer to produce a desired level of suppression. The number of layers is certainly a function of method but typically ranges from a handful to perhaps ten to fifteen.

Finite Difference Boundaries

Sponge

• γ decays exponentially

Perfectly matched layers

• Dispersion applied in each layer Paraxial

• Suppresses waves at angle α_i

$$\frac{l^{2}}{l^{2}} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -\gamma & 1 \\ -\rho v^{2} \nabla \cdot \frac{1}{\rho} \nabla & -\gamma \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$
$$\frac{\partial}{\partial x} \longrightarrow \frac{1}{1 + \frac{i\sigma(x)}{\omega}} \frac{\partial}{\partial x}$$
$$\left\{ \prod_{j=1}^{j=J} \left[(\cos\alpha_{j}) \frac{\partial}{\partial t} - v \frac{\partial}{\partial x} \right] \right\} p = 0$$



Boundaries

Free Surface in an Elastic Medium



A free surface implies that no normal or shear stress are active there, so we can set $\sigma_{i,j,k}^{3,3} = 0$ and $\sigma_{i,j,k}^{1,3} = 0$. The shear stress boundary condition is handled by setting it to zero at z = 0 as well. The vertical stress is not defined at the top boundary but is forced to zero by making the vertical stress antisymmetric for the first two rows above the free surface, i.e., $\sigma_{-1,i}^{3,3} = -\sigma_{0,i,n}^{3,3}$ and $\sigma_{-1,i}^{3,2} = -\sigma_{1,i,n}^{3,3}$

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Questions?



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