Seismic Modeling, Migration and Velocity Inversion

The Partial Differential Wave Equations

Bee Bednar

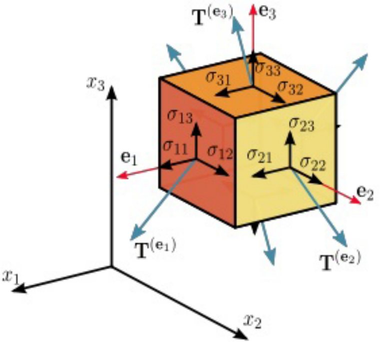
Panorama Technologies, Inc.
14811 St Marys Lane, Suite 150
Houston TX 77079

May 30, 2014

Outline

Outline

3D Stress Equation

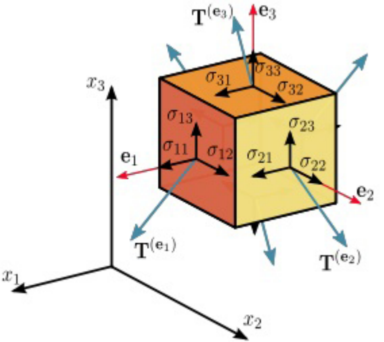


Newton in 3D

$$\frac{\partial^2 u_i}{\partial t^2} = \frac{1}{\rho} \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}.$$

In 3D,the forces that can affect a point are in-line compressional and orthogonal shear. Looking at a small cube each of the nine faces of the cube can move both inward and outward as compressional as well as shear along vertical and horizontal planes.

3D Stress Equation



Newton in 3D

$$\frac{\partial^2 u_i}{\partial t^2} = \frac{1}{\rho} \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}.$$

The stresses σ_{ij} can generated up to three wavefields, $\mathbf{u_i}$. The existence of a wavefield and its strength is completely determined by the properties of the rocks governing propagation. Newton's law relates acceleration to the nine possible forces per unit area (stresses) through the equation above.

3D Hooke

For a linear 3D medium, Hooke's law can be rephrased as

A CHANGE in FORCE per unit volume is equal to the bulk modulus times the increase in volume divided by the original volume.

The 3D stress equation has nine stress factors, σ_{ij}, one for each of the three dimensions and three coupled wavefields, u_i. Hooke's law says that

each component of stress σ_{ij} is linearly proportional to every component of strain E_{mn}

so that

$$\sigma_{ij} = \sum_{m,n} c_{ijmn} E_{mn} = \sum_{m,n} c_{ijmn} \frac{1}{2} \left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m} \right)$$

In this case the c_{ijmn} are elements of what is called the stress tensor.

Coupled Full Elastic Equations

The two equations

$$\frac{\partial^2 u_i}{\partial t^2} = \frac{1}{\rho} \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}$$

$$\sigma_{ij} = \sum_{m,n} c_{ijmn} \frac{1}{2} \left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m} \right)$$

form a coupled system for full elastic wave propagation. Note that superficially there are 81 elements in the stress tensor defined by the c_{ijmn}.

Coupled Full Elastic Equations

If we define

$$v_i = \frac{\partial u_i}{\partial t}$$

then

$$\frac{\partial v_i}{\partial t} = \frac{1}{\rho} \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}$$

and

$$\frac{\partial \sigma_{ij}}{\partial t} = \sum_{m,n} c_{ijmn} \frac{1}{2} \left(\frac{\partial v_m}{\partial x_n} + \frac{\partial v_n}{\partial x_m} \right)$$

form a first-order-in-time-coupled system for full elastic wave propagation. Note that superficially there are 81 elements in the stress tensor defined by the c_{ijmn}.

The $C = [c_{ij}]$ Matrix vs the c_{ijmn} Tensor

Notice that $c_{ijmn} = c_{mnij}$, $c_{ijmn} = c_{ijnm}$, $c_{ijmn} = c_{jimn}$ and $c_{ijmn} = c_{mnij}$, so that after applying the indexing scheme (Voigt scheme)

$$
\begin{array}{cccccccccc}
\textbf{index} & ij & = & 11 & 22 & 33 & 23 & 13 & 12 \\
\textbf{map} & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\textbf{index} & k,l & = & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

one gets

$$
\begin{bmatrix}
c11 & c12 & c13 & c14 & c15 & c16 \\
c12 & c22 & c23 & c24 & c25 & c26 \\
c13 & c23 & c33 & c34 & c35 & c36 \\
c14 & c24 & c34 & c44 & c45 & c46 \\
c15 & c25 & c34 & c45 & c55 & c56 \\
c16 & c26 & c36 & c46 & c56 & c66
\end{bmatrix},
$$

which is the $C = [c_{ij}]$ matrix shown earlier. The symmetry reduces the number of c_{ij} to 21 volumes.

2D Isotropic Elastic Wave Equation

As an example, the 2D Isotropic Elastic Wave Equation is

$$\frac{\partial v_1}{\partial t} = \frac{1}{\rho}\left(\frac{\partial \sigma_{1,1}}{\partial x_1} + \frac{\partial \sigma_{1,3}}{\partial x_3}\right)$$

$$\frac{\partial v_3}{\partial t} = \frac{1}{\rho}\left(\frac{\partial \sigma_{1,3}}{\partial x_1} + \frac{\partial \sigma_{3,3}}{\partial x_3}\right)$$

$$\frac{\partial \sigma_{1,1}}{\partial t} = \frac{\lambda+2\mu}{\rho}\frac{\partial v_1}{\partial x_1} + \frac{\lambda}{\rho}\frac{\partial v_3}{\partial x_3}$$

$$\frac{\partial \sigma_{1,3}}{\partial t} = \frac{\mu}{\rho}\left(\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3}\right)$$

$$\frac{\partial \sigma_{3,3}}{\partial t} = \frac{\lambda+2\mu}{\rho}\frac{\partial v_3}{\partial x_3} + \frac{\lambda}{\rho}\frac{\partial v_1}{\partial x_1}$$

where, in the usual geophysical notation, $x_1 = x$, and $x_3 = z$. Thus, v_1 represents particle velocity in the horizontal and v_3 is particle velocity in the vertical direction. In this case the C matrix is defined by $\lambda + 2\mu$ and μ. Note that these are actually 2D numeric fields. That is, they are 2D functions of x and z.

First Order System

Although the algebra is quite tedious, for any given C matrix, the coupled system in the previous slide can be written as the first order vector system

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{X_1}\frac{\partial \mathbf{v}}{\partial x_1} + \mathbf{X_2}\frac{\partial \mathbf{v}}{\partial x_2} + \mathbf{X_3}\frac{\partial \mathbf{v}}{\partial x_3} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \sigma_{1,1} \\ \sigma_{1,2} \\ \sigma_{1,3} \\ \sigma_{2,2} \\ \sigma_{2,3} \\ \sigma_{3,3} \end{bmatrix}$$

where the elements of the $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$ matrices are determined by the c_{ij} volumes in the C matrix.

First Order System Solution

This latter equation is appealing because it's a one-dimensional-time-domain differential system whose solution is easily expressed as

$$\mathbf{v}(t) = \exp\left[t\mathbf{H}\right]\mathbf{v}(0) + \int_0^t \exp\left[\xi\mathbf{H}\right]\mathbf{S}(t-\xi)d\xi$$

where $\mathbf{v}(0)$ represents the initial conditions, $S(t)$ is the source term and $\mathbf{H}$ is the operator

$$\mathbf{H} = \mathbf{X_1}\frac{\partial}{\partial x_1} + \mathbf{X_2}\frac{\partial}{\partial x_2} + \mathbf{X_3}\frac{\partial}{\partial x_3}$$

Second Order Full Elastic Equation

Substitution of

$$\sigma_{ij} = \sum_{m,n} c_{ijmn} \frac{1}{2} \left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m} \right)$$

into

$$\frac{\partial^2 u_i}{\partial t^2} = \frac{1}{\rho} \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}$$

yields the second order version of the full elastic system

$$\frac{\partial^2 u_i}{\partial t^2} = \sum_{m,n,j} \frac{c_{ijmn}}{\rho} \frac{\partial^2 u_m}{\partial x_n \partial x_j}$$

Second Order Isotropic Elastic Equation

When the C matrix represents a isotropic elastic system, the two shear or transverse waves are identical, so, after considerable algebraic manipulation, one can write

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = (\frac{\lambda + 2\mu}{\rho})\nabla(\nabla \cdot \mathbf{u}) - \frac{\mu}{\rho}\nabla \times \nabla \times \mathbf{u}$$

where the first component of $\mathbf{u} = (u_1, u_3)$ is the compressional wave and the third component is the transverse or shear wave. From a physical viewpoint, the dot product annihilates the compressional component, while the cross product annihilates the shear component.

Second Order Scalar Wave Equation

In a purely acoustic media, the shear parameters are zero, so there is no propagation of shear waves. The 3D elastic equation reduces to the scalar form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\lambda}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

Setting $v = \sqrt{\frac{\lambda}{\rho}}$ produces the traditional scalar wave equation.

Two-Way Wave Equation Summary

In the interest of clarity, the previous derivations were performed under some overly simplistic assumptions. Most notably was the assumption that the density, ρ, was constant as a function of position. Had this not been the case, the full scalar wave equation would have taken the form

$$\frac{\partial^2 p}{\partial t^2} = \rho v^2 \left[\frac{\partial}{\partial x} \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\partial}{\partial z} \frac{1}{\rho} \frac{\partial p}{\partial z} \right].$$

and the fully elastic wave equation would have been a bit more complex. Fortunately, this assumption will not significantly impair out ability to understand the computational aspects of digital wave propagation, so the discussion is continued with the equations as previously derived. The anisotropic models of interest are *VTI*, *TTI*, *ORT*, and *TORT*, all of which are incorporated within the fully elastic wave equation.

Two-Way Wave Equation Summary

There are two fundamental wave equation styles:

Scalar

$$\frac{\partial^2 p}{\partial t^2} = \rho v^2 \left[\frac{\partial}{\partial x} \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\partial}{\partial z} \frac{1}{\rho} \frac{\partial p}{\partial z} \right]$$

and vector

$$\begin{aligned} \frac{\partial \sigma_{ij}}{\partial t} &= \sum_{m,n} c_{ijmn} \frac{1}{2} \left(\frac{\partial v_m}{\partial x_n} + \frac{\partial v_n}{\partial x_m} \right) \\ \frac{\partial v_i}{\partial t} &= \frac{1}{\rho} \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} \end{aligned}$$

Two-Way Wave Equation Summary

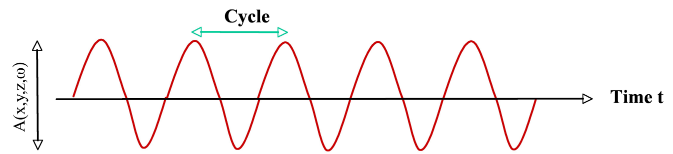
Its worth noting that every seismic wave equation of interest can be derived from the coupled system

$$
\begin{aligned}
\frac{\partial \sigma_{ij}}{\partial t} &= \sum_{m,n} c_{ijmn} \frac{1}{2} \left(\frac{\partial v_m}{\partial x_n} + \frac{\partial v_n}{\partial x_m} \right) \\
\frac{\partial v_i}{\partial t} &= \frac{1}{\rho} \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}
\end{aligned}
$$

so technically this is the only system of concern.

Outline

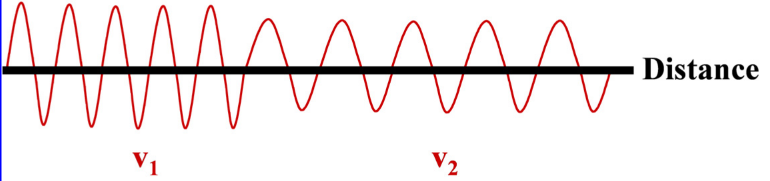
Wavefield Characterization



A monochromatic wavefield in space with frequency f and amplitude $A(x, y, z, f)$ is completely characterized by its

- Frequency f with units of cycles/second
- Angular Frequency $\omega = 2\pi f$ with units of radians/second
- $\frac{v}{f}$ = Wavelength with units of meters/cycle
- or its Wavenumbers
  - Temporal Wave Number $k = \frac{\omega}{v}$ with units of radians/meter
  - x Wavenumber k_x with units of radians/meter
  - y Wavenumber k_y with units of radians/meter
  - z Wavenumber k_z with units of radians/meter

Wavefields in the Earth



A monochromatic wavefield in space at two different velocities. The wavelength and the amplitude change in tandem with velocity changes. When the sinusoid is stretched amplitude reduction maintains the energy level. The vertical wavenumber k_z must change with depth. Wavefield variation with x and/or y also implies that these wavenumbers must also vary.

Fourier Transforms and Wave Equations

Consider the following forms of the constant velocity scalar wave equation

$$\frac{\partial^2 u}{\partial t^2} = v^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad (1)$$

$$\omega^2 U = v^2 \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right) \quad (2)$$

$$\frac{\partial^2 U}{\partial t^2} = -v^2 \left(k_x^2 + k_y^2 + k_z^2 \right) U \quad (3)$$

$$\omega^2 U = -v^2 \left(k_x^2 + k_y^2 + k_z^2 \right) U \quad (4)$$

Equation (1) is called the space-time or XT equation, (2) is call the the frequency-space or FX equation, (3) is the time-wavenumber or TK equation while (4) is normally called the frequency-wavenumber or (FK) equation.

Outline

One-Way Propagation

The PDE

$$\frac{\partial^2 u}{\partial t^2} = v^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

can be written

- In XT as $\frac{\partial^2 u}{\partial z^2} = \frac{1}{V^2}\frac{\partial^2 u}{\partial t^2} - (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$
- In FX as $\frac{\partial^2 U}{\partial z^2} = \frac{\omega^2}{V^2}U - (\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2})$
- In TK as $K_z^2 U = \frac{1}{V^2}\frac{\partial^2 U}{\partial t^2} + (K_x^2 + K_y^2)U$
- In FK as $K_z^2 U = \frac{\omega^2}{V^2}U + (K_x^2 + K_y^2)U$

One-Way Propagation

The PDE

$$\frac{\partial^2 u}{\partial t^2} = v^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

can be written

- In XT as $\frac{\partial^2 u}{\partial z^2} = \frac{1}{V^2} \frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$
- In FX as $\frac{\partial^2 U}{\partial z^2} = \frac{\omega^2}{V^2} U - \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right)$
- In TK as $K_z^2 U = \frac{1}{V^2} \frac{\partial^2 U}{\partial t^2} + (K_x^2 + K_y^2) U$
- In FK as $K_z^2 U = \frac{\omega^2}{V^2} U + (K_x^2 + K_y^2) U$

One-Way Propagation

The PDE

$$\frac{\partial^2 u}{\partial t^2} = v^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

can be written

- In XT as $\frac{\partial^2 u}{\partial z^2} = \frac{1}{V^2}\frac{\partial^2 u}{\partial t^2} - (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$
- In FX as $\frac{\partial^2 U}{\partial z^2} = \frac{\omega^2}{V^2} U - (\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2})$
- In TK as $K_z^2 U = \frac{1}{V^2}\frac{\partial^2 U}{\partial t^2} + (K_x^2 + K_y^2) U$
- In FK as $K_z^2 U = \frac{\omega^2}{V^2} U + (K_x^2 + K_y^2) U$

One-Way Propagation

The PDE

$$\frac{\partial^2 u}{\partial t^2} = v^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

can be written

- In XT as $\frac{\partial^2 u}{\partial z^2} = \frac{1}{V^2}\frac{\partial^2 u}{\partial t^2} - (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$
- In FX as $\frac{\partial^2 U}{\partial z^2} = \frac{\omega^2}{V^2} U - (\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2})$
- In TK as $K_z^2 U = \frac{1}{V^2}\frac{\partial^2 U}{\partial t^2} + (K_x^2 + K_y^2) U$
- In FK as $K_z^2 U = \frac{\omega^2}{V^2} U + (K_x^2 + K_y^2) U$

One-Way Propagation

The PDE

$$\frac{\partial^2 u}{\partial t^2} = v^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

can be written

- In XT as $\frac{\partial^2 u}{\partial z^2} = \frac{1}{V^2}\frac{\partial^2 u}{\partial t^2} - (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$
- In FX as $\frac{\partial^2 U}{\partial z^2} = \frac{\omega^2}{V^2} U - (\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2})$
- In TK as $K_z^2 U = \frac{1}{V^2}\frac{\partial^2 U}{\partial t^2} + (K_x^2 + K_y^2)U$
- In FK as $K_z^2 U = \frac{\omega^2}{V^2} U + (K_x^2 + K_y^2)U$

One-Way Equations in Various Domains

Noticing that the left hand sides appear to be squared terms we can take square roots of both sides and inverse transform over k_z to obtain

$$
\begin{aligned}
\frac{\partial u}{\partial z} &= \pm\sqrt{\left(\frac{1}{V^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)}u \\
\frac{\partial U}{\partial z} &= \pm\sqrt{\left(\frac{\omega^2}{V^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)}U \\
\frac{\partial U}{\partial z} &= \pm i\sqrt{\left(\frac{1}{V^2}\frac{\partial^2}{\partial t^2} + K_x^2 + K_y^2\right)}U \\
\frac{\partial U}{\partial z} &= \pm i\sqrt{\left(\frac{\omega^2}{V^2} + K_x^2 + K_y^2\right)}U
\end{aligned} \tag{5}
$$

One-Way Propagation

In a medium with constant velocity v,

$$w(\mathbf{x}, t) = u(\mathbf{x} - vt) + d(\mathbf{x} + vt)$$

where $\mathbf{x} = (x, y, z)$ is a simple solution to the constant velocity scalar wave equation.

- u represents an upward traveling wave
- d represents a downward traveling wave.

Taking the either the plus or minus sign in one of the equations of the previous slide easily gives us an equation governing either upward (+) or downward (-) propagation. Deriving the equations was relatively easy. Taking the square root is going to represent a serious problem. However we do it we will be doing serious damage to high dip and correct amplitude propagation.

One-Way Equation Summary

- Used for both isotropic and anisotropic propagation
- For full elastic one-way propagation
  - Based on an equation with $v_s = 0$
  - Very non-physical
  - Standing wave noise
  - Lower quality amplitudes and dips

One-Way Propagation

- Why?
  - Increases efficiency
  - Step down one Δz at a time
- Why not?
  - Dips limited to 90 degrees
  - No multiples
  - No Refractions
  - Amplitude distortion

Questions?