# Practical Migration, deMigration, and Velocity Modeling 

Dancing With Waves

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## Outline

## (1) Non Raytrace Methods

- Particle Motion in a Simple 1D Model
- Fundamental Principles
- Newton's Second Law
- Hooke's Law
- The 1D Two-Way Propagation Equation
- Particle Motion in 3D
- Two-Way Wave Equations
- Two-Way Examples
- One-Way Wave Equations
- Applying the Stencils
- Boundary Layers
- Summary


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## A simple 1D model



## The Fundamental Principles



- Particle motion, $u(z, t)$ is governed by two laws
- Newton's second law of motion:
- Force is equal to mass times acceleration
- Hooke's Law
- The amount by which a material body is deformed (the strain) is linearly related to the force causing the deformation (the stress)


## Newton's Second Law



So from Newton's Second Law,

$$
\begin{aligned}
F(z, t)=m a & =m\left(\frac{v(z, t+\Delta t)-v(z, t)}{\Delta t}\right) \\
& =m\left(\frac{\frac{u(z, t+\Delta t)-u(z, t)}{\Delta t}-\frac{u(z, t)-u(z, t-\Delta t)}{\Delta t}}{\Delta t}\right) \\
& =m\left(\frac{u(z, t+\Delta t)-2 u(z, t)+u(z, t-\Delta t)}{\Delta t^{2}}\right)
\end{aligned}
$$

where $a$ is acceleration, $v$ is velocity, and $\Delta t$ is the computational time interval.

## Hooke's Law



Since $F(z, t)$, is determined by the action of the particles on either side of position $z$ Hooke'a Law lets us write

$$
\begin{aligned}
& \begin{aligned}
\begin{aligned}
F(z, t) & =f(z+\Delta z, t)-f(z-\Delta z, t) \\
& =k((u(z+\Delta z, t)-u(z, t))-(u(z, t)-u(z-\Delta z, t))) \\
& =k(u(z+\Delta z, t)-2 u(z, t)+u(z-\Delta z, t))
\end{aligned} \\
\text { where } f(z+\Delta z) \text { and } f(z-\Delta z) \text { are forces from the two } \\
\text { particles surrounding that at } z
\end{aligned} \text {. }
\end{aligned}
$$

## Hooke's Law



After a little algebra

$$
\begin{aligned}
& \frac{u(z+\Delta z, t)-2 u(z, t)+u(z-\Delta z, t)}{\Delta z^{2}}=\frac{\rho}{k} \frac{u(z, t+\Delta t)-2 u(z, t)+u(z, t-\Delta t)}{\Delta t^{2}} \\
& \text { or } \\
& \frac{u(z+\Delta z, t)-2 u(z, t)+u(z-\Delta z, t)}{\Delta z^{2}}=\frac{1}{v^{2}} \frac{u(z, t+\Delta t)-2 u(z, t)+u(z, t-\Delta t)}{\Delta t^{2}}
\end{aligned}
$$

The physical units of $\frac{k}{\rho}$ are $f t^{2} / \sec ^{2}$ so $v=\sqrt{\frac{k}{\rho}}$ is the velocity of propagation.

## The 1D Two-Way Propagation Equation



Thus, the 1D propagator is

$$
\begin{aligned}
u(z, t+\Delta t) & =2 u(z, t)-u(z, t-\Delta t) \\
& +\left(\frac{v \Delta t}{\Delta z}\right)^{2}(u(z+\Delta z, t)-2 u(z, t)+u(z-\Delta z, t))
\end{aligned}
$$

for propagating the particle motion at each time, $t$, to the next at $t+\Delta t$. Note that for any $z$ we must know $u$ at $t$ and $t-\Delta t$ in order to be able to compute the values at $t+\Delta t$.

## Stability

It is worth pointing out that the propagator gives stable results only when

$$
\frac{v \Delta t}{\Delta z} \leq \frac{2}{\pi}<1
$$

## The 1D Two-Way Propagation Equation

- Particles move in both directions
- All forms of motion is allowed
- The amplitude of the motion is correct
- We can compute the motion at any point along the chain
- This provides a trace, $u(z, t)$ at every $z$ on the chain
- $u(z, t)$ is two-dimensional


## Varying $k$



- Nothing in the derivation requires $k$ to be constant
- It can be a function of $z-k(z)$
- In which case $v=v(z)$ also varies as a function of $z$
- Models without lateral velocity change are called $v$ of $z$ models
- Such models have been used to migrate data in time for many years


## 2D/3D Particle Motion



- 2D/3D particle motion is very complex
- Up to three velocities and polarizations
- Each face of the cube or particle can
- compress in or out
- Shear up or down
- Shear right to left
- Velocities are determined by the rocks
- Generally model particle velocity
- Ultimate objective
- Image the entire Earth model
- Including the $C$ matrix
- This is still a really big goal


## A 3D Explicit Finite Difference Propagator

Making the jump from 1D to 3D is not all that difficult, but does require a lot of tedious algebra. In 3D a simple form of the propagating equation is

$$
\begin{aligned}
u(x, y, z, t+\Delta t) & =2 u(x, y, z, t)-u(x, y, z, t-\Delta t) \\
& +\left(\frac{v \Delta t}{\Delta x}\right)^{2} \sum_{k=-K}^{k=K} a_{k} u(x-k \Delta x, y, z, t) \\
& +\left(\frac{v \Delta t}{\Delta y}\right)^{2} \sum_{m=-M}^{m=M} b_{m} u(x, y-m \Delta y, z, t) \\
& +\left(\frac{v \Delta t}{\Delta z}\right)^{2} \sum_{n=-N}^{n=N} c_{n} u(x, y, z-n \Delta z, t) \\
& +s\left(x_{0}, y_{0}, z_{0}, t\right)
\end{aligned}
$$

where the $a_{k}, b_{m}$, and $c_{n}$ coefficients determine the accuracy of the discrete approximation, and $s\left(x_{0}, y_{0}, z_{0}, t\right)$ is the source. Note how closely this resembles the 1D explicit version.

## A 3D Explicit Finite Difference Stencil



Figure: Time volumes at $t$, and $t-\Delta t$ are used to computed the output at time $t+\Delta t$. The stencil surrounds each point in the $t$ volume while only one point is used from $t-\Delta t$ volume. Application of this stencil requires 10 multiplication/sums for each output point. More accurate stencils can require considerably more. Note that the entire volumes at $t$ and $t-\Delta t$ must be computed before the volume at $t+\Delta t$ can be generated. The $\Delta t$ in this case is the computation time increment and has little bearing on recording time.

## Applying the Stencils in Fourier Space

- For each $t$
- For each $x, y$, and $z$
- Fourier Transform
- Calculate coefficients
- Apply coefficients
- Inverse transform
- Next $t=t+\Delta t$
- Large number of XT coefficients
- Very accurate
- Large memory demands
- Large sorting demands

- Considerable memory demands
- Efficient for small data sets
- Not popular
see Kosloff, Dan (Geophysics)


## The 2D Two-Way Propagator at Work



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## The 2D Two-Way Propagator at Work



Figure: A graphic visualization of the 2D propagation process. The 2D propagation begins with values in the blue and red planes filling in values in the green plane using a two-dimensional stencil. The stencil surrounds each point in the $x$, and $z$ directions of the $t$ plane but uses only one value from the $t-\Delta t$ plane. This process proceeds until all values in the $t+\Delta t$ plane have been computed

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## Stability

The factors of the from

$$
\frac{v \Delta t}{\Delta x}, \frac{v \Delta t}{\Delta y} \text {, and } \frac{v \Delta t}{\Delta z}
$$

are extremely important. Assuring that the computations are stable requires that

$$
\Delta t \leq \frac{2}{\pi}\left(\frac{\Delta x_{\min }}{v_{\max }}\right)<1
$$

where $\Delta x_{\min }$ is the smallest of $\Delta x, \Delta y$, and $\Delta z$ and $v_{\max }$ is the maximum velocity in the model.

## 2D Explicit Staggered Grid FD Propagator

Because elastic and particularly anisotropic elastic equations have several additional volumetric parameters the equations themselves are quite complex and very very tedious to derive.


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However, it is worth taking a look at how the calculations progress, but only in 2 D.

## A 2D Staggered Grid Propagator at Work


$\sigma^{1,1} \sigma^{1,2}$


$\sigma^{2,2}$


- Velocity Node
- Shear stress Node
- Normal stress Node

Updated velocities based on stress values in shaded cells while updated stresses use velocity values in shaded cells

Figure: For the five parameter model shown here, points lying on the edge of the shaded areas are on a grid index by $(k+.5) \Delta t$ while those in the center are on a grid index by $k \Delta t$. Staggered grid propagation only requires values from the $t$ plane to calculate values on the $t+\Delta t$ plane.

## A 2D Staggered Grid Propagator at Work



$\sigma^{2,2}$


- Velocity Node
- Shear stress Node
- Normal stress Node

Updated velocities based on stress values in shaded cells while updated stresses use velocity values in shaded cells

Figure: The computational stencil computes the wavefield values on the normal grid from the indicated values on the half and normal grids. Propagation proceeds in much the same manner as discussed for the acoustic propagator.

## Isotropic Elastic Model



(a) Compressional Velocity

(b) Shear Velocity

(c) Density

Figure: Marmousi2. Isotropic elastic version of the original Marmousi data.

## Marmousi2 Snapshots



Figure: Two-dimensional elastic propagation.

## Isotropic Elastic Shot



Figure: Marmousi2 elastic synthetics

## HESS VTI Model


(a) $V_{P}$

(b) $V_{S}$

(c) $\epsilon$

(d) $\delta$

(e) $\rho$

Figure: HESS VTI model in Thomsen notation. Available from the SEG.

## HESS VTI Model

(a) $c_{11}$


(b) $c_{13}$

(c) $C_{33}$

(d) $\mathrm{C}_{55}$

(e) $\rho$

Figure: HESS VTI model in C-matrix notation. Available from the SEG.

## HESS VTI Snapshots



Figure: Anisotropic (VTI) propagation with the HESS VTI model.

## VTI Shot



Figure: Hess-VTI synthetic data.

## Summary

- Two fundamental discrete propagators
- One for scalar equations
- Central differences on regular grid
- Stencil surrounds central point at $t$
- Must compute entire volume at $t$ and $t-\Delta t$ to compute $t+\Delta t$
- One for elastic equations
- Staggered grids
- Five volumes required at each step
- Stencil still surrounds central points on both full and half grid
- Must compute entire volume at $t$ and $t-\Delta t$ to compute $t+\Delta t$


## The 2D One-Way Downward Propagator at Work

Its quite easy to produce a graphical description of a one-way propagator. All one has to do is drop the bottom part of the stencil to produce a one-way downward propagator and drop the top part of the stencil for an upward propagator.


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Figure: Note that for the one-way propagator it is not necessary to compute the entire plane at $t$, and $t-\Delta t$ before computing the plane at $t+\Delta t$. Because values from below the current $z$-slice are excluded, waves travel only in a downward direction. There can be no lateral or upward propagation.

## Graphical Description of 3D One-Way Propagators


(a) Downward only wave propagator

(b) Upward only wave propagator

Figure: One-way propagators in 3D. In (a) downward traveling waves are the result of not using circles from below the current slice. In (b) upward traveling waves are the result of not using circles above the current slice. It is not necessary to calculate the entire previous volume at each step.

## One-Way Propagation

Unfortunately, computing the coefficients for one-way propagators is not straightforward. Development of a one-way propagators requires a change in how the propagation proceeds. Complete solutions, $u(x, y, z, t)$ to our 3D propagation problem can be expressed as,

$$
u(x, y, z, t)=U(x, y, z, t)+D(x, y, z, t)
$$

where $U$ and $D$ are upward only and downward only traveling waves. Its natural question is "How does one find propagating equations for $U$ and $D$.

## One-Way Propagation

We note that if we difference any first order difference, for example

$$
\frac{u(z, t)-u(z-\Delta z, t)}{\Delta z}
$$

we get the second-order difference

$$
\frac{u(z+\Delta z, t)-2 u(z, t)+u(z-\Delta z, t)}{\Delta z^{2}}=\frac{\frac{u(z+\Delta z, t)-u(z, t)}{\Delta z}-\frac{u(z, t)-u(z-\Delta z, t)}{\Delta z}}{\Delta z}
$$

so that in some sense any second order finite difference is a square of a first order difference.

## Differences as Squares

We start with a simple second order finite difference propagator:

$$
\begin{aligned}
& \frac{u(x, y, z+\Delta z, t)-2 u(x, y, z, t)+u(x, y, z-\Delta z, t-\Delta t)}{\Delta z^{2}} \\
& =\frac{1}{v^{2}} \frac{u(x, y, z, t+\Delta t)-2 u(x, y, z, t)+u(x, y, z, t-\Delta t)}{\Delta t^{2}} \\
& -\frac{u(x+\Delta x, y, z, t)-2 u(x, y, z, t)+u(x-\Delta x, y, z, t-\Delta t)}{\Delta x^{2}} \\
& -\frac{u(x, y+\Delta, z, t)-2 u(x, y, z, t)+u(x, y-\Delta y, z, t-\Delta t)}{\Delta y^{2}}
\end{aligned}
$$

## Algebraic Representations

If, in the space-time domain, we let

$$
\begin{aligned}
& T^{2}=\frac{(u(x, y, z, t+\Delta t)-2 u(x, y, z, t)+u(x, y, z, t+\Delta t))}{\Delta t^{2}} \\
& z^{2}=\frac{u(x, y, z+\Delta z, t)-2 u(x, y, z, t)+u(x, y, z-\Delta z, t)}{\Delta z^{2}} \\
& X^{2}=\frac{(u(x+\Delta x, y, z, t)-2 u(x, y, z, t)+u(x-\Delta x, y, z, t))}{\Delta x^{2}} \\
& Y^{2}=\frac{(u(x, y+\Delta y, z, t)-2 u(x, y, z, t)+u(x, y+\Delta y, z))}{\Delta y^{2}}
\end{aligned}
$$

we get the XT form $Z^{2}=\frac{T^{2}}{v^{2}}-\left(X^{2}+Y^{2}\right)$ so that

$$
Z=\frac{u(x, y, z+\Delta z, t)-u(z, y, z, t)}{\Delta z}= \pm \sqrt{\frac{T^{2}}{v^{2}}-\left(X^{2}+Y^{2}\right)}
$$

## Algebraic Representations

Transforming over space and time,

$$
\begin{aligned}
& \frac{T^{2}}{v^{2}} \leftrightarrow k^{2}=\frac{\omega^{2}}{v^{2}} \\
& Z^{2}
\end{aligned} \leftrightarrow k_{z}^{2} .
$$

Produces similar forms

- FK $-k_{z}^{2}=k^{2}-\left(k_{x}^{2}+k_{y}^{2}\right)$
- $\mathrm{FX}-Z^{2}=k^{2}-\left(X^{2}+Y^{2}\right)$
- $\mathrm{KT}-k_{z}^{2}=\frac{T^{2}}{V^{2}}-\left(k_{x}^{2}+k_{y}^{2}\right)$


## Two For the Price of One

Note that the every one of these formulas is actually two equations in one. For example, in space-time,

$$
Z=+\sqrt{\frac{1}{v^{2}} T^{2}-\left(X^{2}+Y^{2}\right)}
$$

is an equation for upward traveling waves while the other

$$
\begin{equation*}
Z=-\sqrt{\frac{1}{v^{2}} T^{2}-\left(X^{2}+Y^{2}\right)} \tag{1}
\end{equation*}
$$

is for downward traveling waves. To utilize either requires finding an approximation for the square root on the right hand side. This is true for all of the forms described above.

## Square Roots

There are two well known approaches for taking the square roots. One is a standard formula for finding the square root of an arbitrary number. In space-time the approximation is:

$$
\begin{equation*}
Z= \pm \frac{T^{2}}{v^{2}} \sqrt{1.0-\frac{\left(X^{2}+Y^{2}\right) v^{2}}{T^{2}}} \approx \pm \frac{T}{v}-\frac{4 \frac{T^{2}}{v^{2}}-3\left(X^{2}+Y^{2}\right)}{4 \frac{T^{2}}{v^{2}}-\left(X^{2}+Y^{2}\right)} \tag{2}
\end{equation*}
$$

## Square Root Approximations

The other approach uses a Taylor series approximation to reduce the square form into a usable equation. For a reference slowness $\frac{1}{v_{0}(z)}$ and $\Delta s=s-s_{0}$ the square root of $k_{z}^{2}$ can be written:

$$
k_{z}= \pm\left(\sqrt{k_{0}^{2}-k_{x}^{2}-k_{y}^{2}}+\omega \Delta s+\frac{2\left(k_{x}^{2}+k_{y}^{2}\right)}{4 k_{0}^{2}-3\left(k_{x}^{2}+k_{y}^{2}\right)} \omega \Delta s^{2}\right)
$$

where $k_{0}=\frac{\omega}{v_{0}}$ and we have ignored terms of higher order then 2 . Similar approximations can be written for the remaining FX and KT forms.

## Using the Square Root Approximations

Each of the preceding square root approximations is used in different ways in 2D, replacing $T^{2}, Z$, and $X^{2}$ in the XT form with differences yields

$$
\begin{aligned}
u(x, z & +\Delta z, t+\Delta t)=u(x, z, t+\Delta t) \\
+ & \frac{u(x, z, t+\Delta t)-u(x, z, t)}{v \Delta t} \\
& -\frac{4\left(\frac{u(x, z, t+\Delta t)-2 u(x, z, t)+u(x, z, t-\Delta t)}{v^{2} \Delta t^{2}}\right)^{2}-3\left(\frac{u(x+\Delta x, z, t)-2 u(x, z, t)+u(x-\Delta x, z, t)}{\Delta x^{2}}\right)^{2}}{4\left(\frac{u(x, z, t+\Delta t)-2 u(x, z, t)+u(x, z, t-\Delta t)}{v^{2} \Delta t^{2}}\right)^{2}-\left(\frac{u(x+\Delta x, z, t)-2 u(x, z, t)+u(x-\Delta x, z, t)}{\Delta x^{2}}\right)^{2}}
\end{aligned}
$$

Solving this for $u(x, z+\Delta z, t+\Delta t)$ necessitates clearing fractions along with a considerable amount of algebraic manipulation.

## Using the Square Root Approximations

After doing that, a lengthy algebraic manipulation allows us for a fixed $z+\Delta z$ to write the preceding equation in the matrix form

$$
\mathbf{A} u(x, z+\Delta z, t)=\mathbf{B} u(x, z, t)
$$

so that

$$
u(x, z+\Delta z, t)=\mathbf{A}^{-1} \mathbf{B} u(x, z, t)
$$

Where $\mathbf{A}$ and $\mathbf{B}$ are derived from the finite differences and the underlying Earth model. This matrix approach is said to be an implicit stencil method because the actual stencil coefficients are determined from the inverse $\mathbf{A}^{-1}$ and $B$.

## Square Root Approximations

The process described by the last equations in the previous slide is said to be an implicit propagator. The word implicit derives from the fact that one has to perform a matrix inversion for each downward $\Delta z$ step. While it can be done fairly accurately, inverting $A$ in 3D is not an easy and consequently the methodology has not gained the acceptance it probably deserves. Consequently researches sought more efficient and easier methods through alternative approaches.

## Square Root Approximations

Another approach to taking the square root takes advantage of Fourier domain simplifications. Transforming over over both time, $t$, and space, $(x, y)$, produces the simple frequency-wavenumber multiplication propagator,

$$
\begin{aligned}
U\left(k_{x}, k_{y}, z+\Delta z, \omega\right) & =\exp \left(+i k_{z} \Delta z\right) U\left(k_{x}, k_{y}, z, \omega\right) \\
D\left(k_{x}, k_{y}, z+\Delta z, \omega\right) & =\exp \left(-i k_{z} \Delta z\right) D\left(k_{x}, k_{y},, z, \omega\right)
\end{aligned}
$$

where

$$
\begin{equation*}
k_{z}= \pm \sqrt{\frac{\omega^{2}}{v^{2}}-k_{x}^{2}-k_{y}^{2}} \tag{3}
\end{equation*}
$$

and $v=v(x, y, z)$ is the velocity in the interval between $z$ and $z+\Delta z$. There is no doubt this is a great simplification but the square root problem remains and looks very similar to the previous case.

## Square Root Approximations

Using the first three terms of the series provides the expression

$$
\exp \left( \pm k_{z} \Delta z\right)=\exp \left( \pm i \sqrt{k_{0}-k_{x}^{2}-k_{y}^{2}} \Delta z\right) \exp ( \pm i \omega \Delta s \Delta z) \exp \left( \pm i \frac{2\left(k_{x}^{2}+k_{y}^{2}\right)}{4 k_{0}^{2}-3\left(k_{x}^{2}+k_{y}^{2}\right)} \omega \Delta s^{2} \Delta z\right)
$$

for the exponential. Each of the terms in the exponential in the previous slide gives rise to a new modeling algorithm. Inclusion of interpolation generates two more.

## Phase-Shift

$\qquad$
$\qquad$
$\qquad$ $\mathbf{V}(\mathbf{z})$
$\qquad$
$\qquad$
$\qquad$
$\qquad$
$\qquad$

- The 1st, based on $\exp \left( \pm i \sqrt{k_{0}-k_{x}^{2}-k_{y}^{2}} \Delta z\right)$ is phase shift modeling
- Applied only in FK
- Assumes that the velocity varies only vertically


## Phase-Shift-Plus-Interpolation


$\qquad$
$\qquad$
$\qquad$

- The 2nd, based on interpolating several phase shifts, is phase-shift-plus interpolation (PSPI)
- First extension to full 3D velocity variation
- Applied only in FK
- Difficult to do the interpolation accurately


## Split-Step

## $\mathbf{V}(\mathbf{x}, \mathbf{y}, \mathbf{z})$

- The 3rd, based on $\exp ( \pm i \omega \Delta s \Delta z)$ after phase shift is split-step modeling
- Second extension to full 3D velocity variation
- Applied in FK and then in FX
- Removed interpolation issue
- Was shown to be too inaccurate


## Extended Split-Step


$\qquad$
$\qquad$
$\qquad$
$\qquad$

- The 4th, using interpolation after split-step, is extended split step
- Third extension to full 3D velocity variation
- Applied only in FK
- Difficult to do the interpolation accurately


## Phase-Screen

## $\mathbf{V}(\mathbf{x}, \mathrm{y}, \mathrm{z})$

$\qquad$
$\qquad$
$\qquad$
$\qquad$
$\qquad$

- The 5th, based on $\exp \left( \pm i \frac{2\left(K_{x}^{2}+k_{y}^{2}\right)}{44_{0}^{2}-3\left(k_{x}^{2}+k_{y}^{2}\right)} \omega \Delta s^{2} \Delta z\right)$ after split step is phase screen modeling
- Applied in FK then in FX, and finally in FK.
- The ratio $\frac{2\left(k_{x}^{2}+k_{y}^{2}\right)}{4 k_{0}^{2}-3\left(k_{x}^{2}+k_{y}^{2}\right)}$ means the phase-screen method is implicit.
- Many different implementation because of the implicit nature.


## Adding Another Bounce

- Adding another bounce
- Downward propagate to max depth
- Upward propagate from max to min depth
- Some two-way propagation
- Increased dip response
- Does not achieve all directions
- Used as a modeling scheme
- Inaccurate amplitudes

After Claerbout 1984


## Wave Equations and Stencils

- All so called Wave Equation methods are stencil based
- There is always an equivalent set of XT coefficients
- The number of coefficients is usually greater then those used in FD schemes
- Coefficients are calculated in the domain in which they are applied
- Space-Time (XT)
- Space-Frequency (XF)
- Frequency-Wavenumber (FK)
- Wavenumber-Time (K-T)
- But, this is rare to non-existent
- Wavenumber-Frequency-Space (FKX)
- Dual-domain methods
- One-way methods require approximation of the original two-way equation
- By taking a square root of derivatives
- This is their Achilles heel


## Applying the Stencils

- It's usually assumed that FD is applied in the XT domain
- But this is certainly not necessary
- They can be applied in any combination of Fourier and XT domains
- Fourier transform over time to the frequency domain (FX)
- Fourier transform over space to the wavenumber domain (TK)
- Fourier transform to both frequency and wavenumber (FK)
- Fourier transform back to the XT domain
- Each step can be applied in multiple domains


## Boundaries



Figure: Realistic seismic simulations generally include procedures for suppressing boundary reflections. Modern approaches begin by surrounding the model with a small number of fake layers. Modified equations for absorbing energy are then applied layer by layer to produce a desired level of suppression. The number of layers is certainly a function of the particular method but typically ranges from a handful to perhaps ten to fifteen.

## Free Surfaces



Figure: A free surface is one in which no normal or shear stress are active. Thus, we can set the normal and horizontal (shear) stresses to zero there. Such surfaces are characteristically the boundary between two homogeneous liquids and the best geophysical example is the boundary between air and water. Since we can turn the free surface on and off as we choose, we can generate synthetic data with or without free surface multiples.

## Summary

- Two-way and One-way modeling
- Foundation for what has been referred to as wave equation methods
- Fact is that all migration methods are wave equation based.
- The most prominent of the migration methods are
- Reverse-time-migration (RTM)
- Wave-equation-migration (WEM)
- Usually phase-screen style
- PSPI and Extended Split-Step are still used


## Summary

- Accuracy hierarchy (Decreases left to right)
- RTM $\rightarrow$ WEM
- WEM issues
- Accuracy of the square root approximation.
- Amplitude inaccuracies
- Sensitivity to strong lateral velocity variation
- RTM issues
- If implemented properly, none
- Velocity sensitivity (Decreases left to right)
- WEM $\rightarrow$ RTM
- Computational efficiency (Decreases left to right)
- WEM $\rightarrow$ RTM $\rightarrow$ GB


## Questions?

